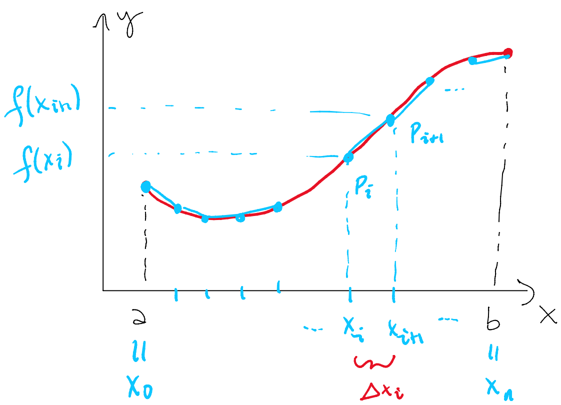
# Arc length and surface area

We will learn how to find the arclength of a curve defined by $y=f(x)$, $a\le x\le b$ for "nice" functions $f$.

Let $f$ be such that $f'$ is continuous on $[a,b]$. Let $a=x_0<x_1<\dots<x_n=b$ be a partition of $[a,b]$. Let $P_i=(x_i,f(x_i))$. Then the total length of the line segments joining $P_i$ to $P_{i+1}$ is



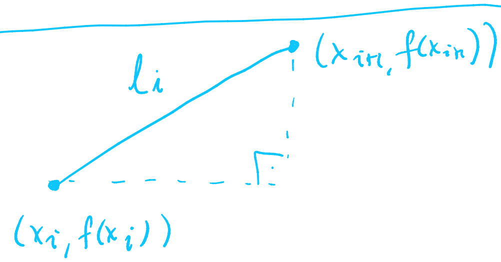

$$\sum_{i=0}^{n-1}|P_iP_{i+1}| =$$

$$\sum_{i=0}^{n-1}\sqrt{1+\left(\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right)^2}\,\Delta x_i =$$

$$\sum_{i=0}^{n-1}\sqrt{1+f'(x_i^*)^2}\,\Delta x_i$$

for some $x_i\le x_i^*\le x_{i+1}$, by MVT, we have

$$\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}=f'(x_i^*)$$




$$l_i=\sqrt{\underbrace{(x_{i+1}-x_i)^2}_{\Delta x_i}+(f(x_{i+1})-f(x_i))^2}$$

$$l_i=\sqrt{\Delta x_i^2\left(1+\left(\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right)^2\right)}$$

Observe that this sum is a general Riemann sum for $\sqrt{1+f'(x)^2}$ over $[a,b]$. As $n\to\infty$ and $\max\Delta x_i\to 0$, these Riemann sums approach to

$$\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

which is equal to the arclength of the graph of $f(x)$ over $[a,b]$.

**Example:** Find the length of the curve defined by $y=\frac{x^2}{8}-\ln(x)$, $1\le x\le 3$.

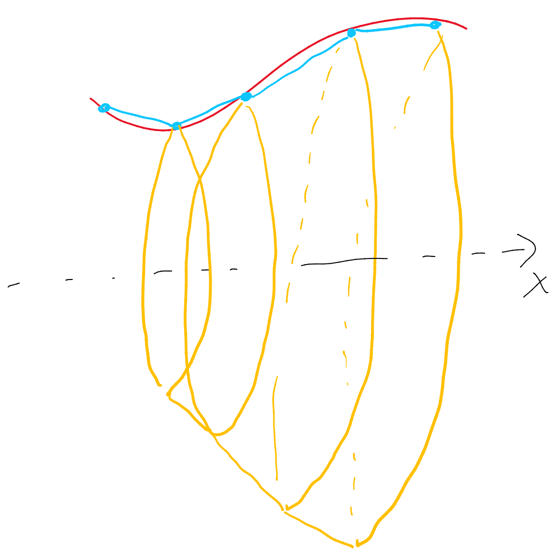
**Solution:** This curve is the graph of $f(x) = \frac{x^2}{8} - \ln(x)$ over $[1,3]$. So its length is

$$\int_1^3 \sqrt{1+(f'(x))^2}\,dx = \int_1^3 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx = \int_1^3 \sqrt{1+\frac{x^2}{16}+\frac{1}{x^2}-\frac{2\cancel{x}}{4}\frac{1}{\cancel{x}}}\,dx$$

$$= \int_1^3 \sqrt{\frac{x^2}{16}+\frac{1}{x^2}+\frac{1}{2}}\,dx = \int_1^3 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx = \int_1^3 \left|\frac{x}{4}+\frac{1}{x}\right|dx = \int_1^3 \frac{x}{4}+\frac{1}{x}\,dx$$

$$= \left(\frac{x^2}{8}+\ln(x)\right)\Bigg|_1^3 = \left(\frac{9}{8}+\ln(3)\right) - \frac{1}{8} = 1+\ln(3)$$

Suppose that we wish to compute the surface area of the solid (in space) obtained by rotating the graph $f$ over $[a,b]$, where $f$ is such that $f'$ is continuous over $[a,b]$, about the x-axis. By creating polygonal approximations as before and rotating these about the x-axis and considering the total surface area of the "strips", we can get an approximation for the surface area of this object.

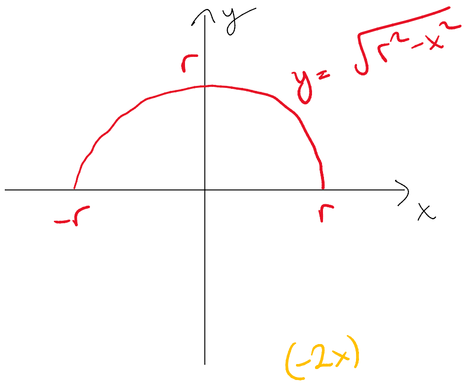



As before, if we take the limit as $n \to \infty$ and $\max \Delta x_i \to 0$, we see that the surface area of the solid obtained by rotating $y=f(x)$, $a \le x \le b$ about the x-axis is

$$\int_a^b 2\pi\,|f(x)|\,\sqrt{1+(f'(x))^2}\,dx$$

**Example:** Show that the surface area of a sphere with radius $r$ is $4\pi r^2$

**Solution:** let $f(x) = \sqrt{r^2 - x^2}$. Then the solid obtained by rotating $y = f(x)$, $-r \le x \le r$ is a sphere with radius $r$. So its surface area is




$$f'(x) = \frac{1}{2\sqrt{r^2-x^2}} \cdot (-2x)$$

$$\int_{-r}^{r} 2\pi \left|\sqrt{r^2-x^2}\right| \sqrt{1 + \frac{4x^2}{4(r^2-x^2)}}\, dx =$$

$$\int_{-r}^{r} 2\pi \left|\sqrt{r^2-x^2}\right| \sqrt{\frac{4(r^2-\cancel{x^2}) + \cancel{4x^2}}{4(r^2-x^2)}}\, dx =$$

$$\int_{-r}^{r} \cancel{2}\pi \, \cancel{\sqrt{r^2-x^2}} \, \frac{2r}{\cancel{2\left|\sqrt{r^2-x^2}\right|}}\, dx = \int_{-r}^{r} 2\pi r\, dx = 2\pi r x \Big|_{-r}^{r}$$

$$= 2\pi r^2 - (-2\pi r^2) = 4\pi r^2$$

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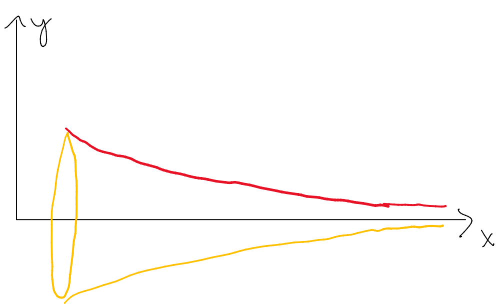
**For fun:** Consider the solid obtained by rotating $y = \frac{1}{x}$ over $[1, \infty)$ about the x-axis. The resulting object is called Gabriel's horn or Torricelli's trumpet.

The surface area of this object is




$$\int_{1}^{\infty} 2\pi \frac{1}{x} \underbrace{\sqrt{1 + \left(\frac{-1}{x^2}\right)^2}}_{\ge 1} dx > \int_{1}^{\infty} 2\pi \frac{1}{x}\, dx = +\infty$$

On the other hand, the volume of this solid is

$$\int_{1}^{\infty} \pi \frac{1}{x^2}\, dx = \lim_{R\to\infty} \int_{1}^{R} \pi \frac{1}{x^2}\, dx = \lim_{R\to\infty} \pi\left(\frac{-1}{x}\right)\Big|_{1}^{R} = \lim_{R\to\infty} \cancelto{0}{\frac{-\pi}{R}} + \pi = \pi$$

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